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Two-dimensional quantum gravity and quasiclassical integrable hierarchies*

A Marshakov

Theory Department, P N Lebedev Physics Institute, Institute of Theoretical and Experimental Physics, Moscow, Russia

E-mail: mars@lpi.ru and mars@itep.ru

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Abstract
The main results for the two-dimensional quantum gravity, conjectured from the matrix model or integrable approach, are presented in the form to be compared with the worldsheet or Liouville approach. In a spherical limit, the integrable side for minimal string theories is completely formulated using simple manipulations with two polynomials, based on residue formulae from quasiclassical hierarchies. Explicit computations for particular models are performed and certain delicate issues of nontrivial relations among them are discussed. They concern the connections between different theories, obtained as expansions of basically the same stringy solution to dispersionless Kadomtsev–Petviashvili hierarchy in different backgrounds, characterized by nonvanishing background values of different times, being the simplest known example of the change of the quantum numbers of physical observables, when moving to a different point in the moduli space of the theory.

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1. Introduction

The problem of solving two-dimensional quantum gravity has already existed for more than 20 or even more than 25 years. By its basic definition one usually takes the Polyakov path integral [1], where the integration over the metrics on two-dimensional string worldsheets has been reduced to the study of naively simple, but in fact quite nontrivial, two-dimensional conformal Liouville field theory. The worldsheet approach allowed us to immediately determine only the relatively simple quantities—such as scaling dimensions—of the operators of the two-dimensional quantum gravity [2, 3]. The computation of their correlators—even on the

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worldsheets of the simplest spherical topology—appeared to be a problem of much higher complexity, and was (yet only partially) solved very recently.

Fortunately, two-dimensional quantum gravity is a *renormalizable* theory—in the most physically important sense of the word, it means that the details of regularization of the theory at the microscopic scale do not affect its macroscopic properties: the 'observable' scaling dimensions and correlators. In other words, two-dimensional quantum gravity possesses a strong *universality* property which means that quite different methods of the computation give rise basically to the same result.

The first sign of this was already observed in the middle of the 1980s of the last century. The idea of summing over the discrete triangulations of worldsheets instead of the integrations over the metric in continuous theory had demonstrated its efficiency in a two-dimensional case, quite in contrast with the nonrenormalizable gravity of higher dimensions. Moreover, it turned out that summing over triangulations of the two-dimensional surfaces can be itself reformulated as summing over the fat graphs of the matrix models [4]. The duality between the matrix model (the zero-dimensional gauge theory) and continuous two-dimensional worldsheet gravity is in fact nothing but the first studied example of the nowadays famous gauge/string duality.

By the matrix model approach, two-dimensional quantum gravity was claimed to be 'completely solved' [5] in the beginning of the 1990s of the previous century. This solution was nicely formulated [6, 7] in terms of special stringy solutions to the hierarchies of integrable equations, they being all the well-known polynomial reductions of the Kadomtsev–Petviashvili (KP) hierarchy. In practice, this has opened a possibility of computing exactly the correlators in two-dimensional gravity (in the framework of the 'matrix model' approach) at least in the spherical approximation (when all closed string loops are suppressed) by methods of the dispersionless KP (dKP) hierarchy, which turn this problem into the problem of solving algebraic nonlinear equations. Below, following [8], we shall demonstrate how this leads straightforwardly to the computation of invariant correlation numbers—the ratios of the correlation functions which do not depend upon the normalizations of particular operators.

However, it is still a great puzzle and, at least partially, an open problem whether the matrix model approach leads *exactly* to the same results as the original worldsheet approach. Partially this is related to the fact that the worldsheet quantum Liouville theory of [1] is a rather specific two-dimensional quantum field theory which is yet to be fully understood. The two- and three-point functions in Liouville theory were computed in the early 1990s [9, 10], but it turned out that it was only after the discovery of the higher order equations of motion by Alesha Zamolodchikov [11] that it appeared to be possible to compute the generic multipoint correlation functions of the operators of minimal (p, q) models coupled to the two-dimensional Liouville gravity, where the integrands on the moduli spaces of worldsheets with punctures are basically reduced after using the higher order equations to the total derivatives.

These correlation functions could now be compared with the results extracted from the 'matrix model' approach or, more strictly, from the formulation of minimal string theory in the language of integrable hierarchies.

2. dKP for (p, q)-critical points

According to a widely believed hypothesis, the so-called (p, q) critical points of the two-dimensional gravity (or (p, q) minimal string theory) are most effectively described by the tau function of the p-reduced KP hierarchy, satisfying the string equation. The logarithm of this tau function should be further expanded around certain background values of the time variables, with necessary $t_{p+q} \neq 0$. In particular, it means that the correlators on the worldsheets of spherical topology (the only ones, partially computed by now by means of two-dimensional conformal field theory [11, 12]) are governed by the quasiclassical tau function of the dispersionless KP or dKP hierarchy, which is a very reduced case of generic quasiclassical hierarchy from [13].

For each (p, q) th minimal theory one should consider a solution of the p-reduced dKP hierarchy or, more strictly, its expansion in the vicinity of nonvanishing $t_{p+q} = \frac{p}{p+q}$ and vanishing other times, perhaps except for the cosmological constant x, chosen in a different way for the different theories (the so-called conformal backgrounds). If $q = p+1$ (the unitary series) the cosmological constants $x \sim t_1$ basically coincide with the main first time of the KP hierarchy, but for 'non-unitary' backgrounds the quantum numbers change, and this causes certain nontrivial relations on the space of KP solutions to be discussed below.

2.1. Residue formulae

The geometric formulation of results for minimal string theories in terms of the quasiclassical hierarchy can be sketched in the following way.

- For each (p, q)th point, take a pair of polynomials

$$X = \lambda^p + \cdots \qquad Y = \lambda^q + \cdots \tag{1}$$

of degrees p and q respectively. They can be thought of as a dispersionless version of the Lax and Orlov–Shulman operators of the KP theory,

$$\begin{aligned} &[\hat{X}, \hat{Y}] = \hbar \\ &\hat{X} = \partial^p + \cdots, \qquad \hat{Y} = \partial^q + \cdots, \end{aligned} \tag{2}$$

or as a pair of (here already integrated) Krichever differentials with the fixed periods on a complex curve (for dKP—a rational curve with global uniformizing parameter λ). It is also convenient to combine these polynomials into a generating differential

$$\mathrm{d}S = Y\,\mathrm{d}X, \tag{3}$$

whose periods and singularities define the variables of the quasiclassical hierarchy. Since on the rational curve (λ-plane or Riemann sphere with the marked point P_0, where $\lambda = \infty$) all periods of (3) vanish, the time variables are related to the residues or the singular part of expansion of differential $\mathrm{d}S$ at point P_0.

- The variables of the dispersionless KP hierarchy are therefore introduced by residue formulae [13–15]

$$\begin{aligned} t_k &= \frac{1}{k}\mathrm{res}_{P_0}\xi^{-k}\,\mathrm{d}S, \qquad k > 0 \\ \frac{\partial \mathcal{F}}{\partial t_k} &= \mathrm{res}_{P_0}\xi^{k}\,\mathrm{d}S, \qquad k > 0, \end{aligned} \tag{4}$$

where

$$\xi = X^{\frac{1}{p}} = \lambda\left(1 + \cdots + \frac{X_0}{\lambda^p}\right)^{\frac{1}{p}} \tag{5}$$

is the distinguished inverse local co-ordinate at the point P_0, where $\lambda(P_0) = \infty$ and $\xi(P_0) = \infty$. From (4), it also follows for the second derivatives

$$\frac{\partial^2 \mathcal{F}}{\partial t_n \partial t_k} = \mathrm{res}_{P_0}(\xi^k\,\mathrm{d}H_n) \tag{6}$$

while the third derivatives are given by the formula

$$\frac{\partial^3 \mathcal{F}}{\partial t_k \partial t_l \partial t_n} = \mathrm{res}_{\mathrm{d}X=0}\left(\frac{\mathrm{d}H_k\,\mathrm{d}H_l\,\mathrm{d}H_n}{\mathrm{d}X\,\mathrm{d}Y}\right). \tag{7}$$

In (6) and (7), the set of 1-forms

$$dH_k = \frac{\partial dS}{\partial t_k}, \qquad k \geqslant 1 \tag{8}$$

(derivatives are taken at fixed X), corresponds to the dispersionless limit of the KP flows and can be integrated up to polynomial expressions

$$H_k = X(\lambda)_+^{k/p} \tag{9}$$

in uniformizing co-ordinate $\lambda = H_1$.

Also note that the tau functions of (p, q) and (q, p) theories do not coincide, but are related by the Legendre or Fourier transform [16], exchanging the polynomials (1) by each other $X \leftrightarrow Y$.

2.2. Solution to dKP

The fact that 1-forms (8) can be integrated up to polynomials (9) leads to an explicit expression for the integrated generating differential (3), or

$$S = \sum_{k=1}^{p+q} t_k H_k = \sum_{k=1}^{p+q} t_k X^{k/p}(\lambda)_+, \qquad k \bmod p, \tag{10}$$

depending already upon the coefficients of the polynomial $X(\lambda)$ only. In other words, formula (10) means that the first part of equations (4) has been already effectively resolved for the coefficients of $Y(\lambda)$. The dependence of the coefficients of $X(\lambda) = \lambda^p + \sum_{k=0}^{p-2} X_k \lambda^k$ over the KP times (4) is determined in the most easy way from $dS|_{dX=0} = 0$, which is now a system of $p-1$ 'hodograph' equations $\frac{dS}{d\lambda} = 0$ imposed at $p-1$ roots of $X'(\lambda) = 0$.

A simple observation that any Hamiltonian (9) is a polynomial in terms of the variable $\lambda = H_1$ leads to dispersionless Hirota equations, which express any second derivative $\frac{\partial^2 \mathcal{F}}{\partial t_k \partial t_n}$ with arbitrary k and l in terms of the second derivatives $\frac{\partial^2 \mathcal{F}}{\partial t_k \partial t_1}$ where one of the indices is fixed and corresponds to the first time. From formulae (4), one finds that

$$dS \underset{\xi\to\infty}{=} \sum \left(k t_k \xi^{k-1} d\xi + \frac{\partial \mathcal{F}}{\partial t_k} \frac{d\xi}{\xi^{k+1}} \right), \tag{11}$$

which is just an expansion in the local co-ordinate at the marked point P_0. Taking the time derivatives (cf with (8)) gives the set

$$H_k = \frac{\partial S}{\partial t_k} = \xi^k - \sum_j \frac{\partial^2 \mathcal{F}}{\partial t_k \partial t_j} \frac{1}{j \xi^j} = \xi^k(\lambda)_+, \qquad k > 0, \tag{12}$$

which forms a basis of meromorphic functions with poles at the point P_0 or just a particular polynomial basis, explicitly fixed by the last equation. The set of the powers λ^k has the same singularities as the set of functions (12), i.e. these two are related by simple linear transformation, e.g.

$$\begin{aligned} &H_1 = \lambda, \qquad H_2 = \lambda^2 + 2\frac{\partial^2 \mathcal{F}}{\partial t_1^2}, \\ &H_3 = \lambda^3 + 3\frac{\partial^2 \mathcal{F}}{\partial t_1^2}\lambda + \frac{3}{2}\frac{\partial^2 \mathcal{F}}{\partial t_1 \partial t_2}, \ldots. \end{aligned} \tag{13}$$

These equalities follow from the comparison of the singulars at the P_0 part of their expansions in ξ, following from (12). Comparing the negative 'tails' of the expansion in ξ of both sides of equation (13) expresses derivatives $\frac{\partial^2\mathcal{F}}{\partial t_k \partial t_l}$ (of H_k on the lhs) in terms of only those with $k = 1$ (of $\lambda = H_1$ on the rhs). These relations are called the dispersionless KP or the dKP Hirota equations, e.g.

$$\frac{\partial^2\mathcal{F}}{\partial t_1 \partial t_3} = \frac{3}{8}X_0^2 = \frac{3}{2}\left(\frac{\partial^2\mathcal{F}}{\partial t_1^2}\right)^2, \qquad \frac{\partial^2\mathcal{F}}{\partial t_3 \partial t_3} = \frac{3}{8}X_0^3 = 3\left(\frac{\partial^2\mathcal{F}}{\partial t_1^2}\right)^3. \tag{14}$$

We have listed here those that will be of some interest for two-dimensional quantum gravity.

2.3. Scaling

Under the scaling $X \to \Lambda^p X$, $Y \to \Lambda^q Y$ (induced by $\lambda \to \Lambda\lambda$ and therefore $\xi \to \Lambda\xi$), the times (4) transform as $t_k \to \Lambda^{p+q-k} t_k$. Then from the second formula of (4) it follows that the function $\mathcal{F}$ scales as $\mathcal{F} \to \Lambda^{2(p+q)}\mathcal{F}$ or, for example, as

$$\mathcal{F} \propto t_1^{2\frac{p+q}{p+q-1}} f(\tau_k), \tag{15}$$

where f is supposed to be a scale-invariant function of the corresponding dimensionless ratios of the times $\tau_k = t_k \big/ t_1^{\frac{p+q-m}{p+q-1}}$ (4). In the simplest $(p, q) = (2, 2K - 1)$ case of dispersionless KdV, one also expects a natural scaling of the form

$$\mathcal{F} \propto (t_{2K-3})^{K+\frac{1}{2}}\,\mathsf{f}(\mathsf{t}_l) \tag{16}$$

with $\mathsf{t}_l = t_{2l-1}/(t_{2K-3})^{(K-l+1)/2}$, where the role of the cosmological constant is played by the time $t_{2K-3} \propto \Lambda^4$.

2.4. KdV series

More explicit formulae can be written for the 'KdV series' $(p, q) = (2, 2K-1)$, corresponding to the $p = 2$ KdV reduction of the KP hierarchy. Now

$$\begin{aligned} X &= \lambda^2 + 2u, \qquad \xi = \sqrt{X} = \sqrt{\lambda^2 + 2u} \\ Y &= \lambda^{2K-1} + \sum_{k=1}^{K-1} y_k \lambda^{2k-1} \end{aligned} \tag{17}$$

and the explicit formula (10) reads as

$$S = \sum_{k=1}^{K+1} t_{2k-1} X^{k-1/2}(\lambda)_+. \tag{18}$$

Dependence on u upon the flat times is determined by a *single* equation

$$\mathrm{d}S|_{\mathrm{d}X=0} = 0 \tag{19}$$

since $\mathrm{d}X = 2\lambda\,\mathrm{d}\lambda$ has the only zero at $\lambda = 0$, or vanishing of the polynomial

$$P(u) \equiv \frac{1}{2}\frac{\mathrm{d}S}{\mathrm{d}\lambda}\bigg|_{\lambda=0} = \sum_{k=0}^{K} \frac{(2k+1)!!}{k!} t_{2k+1} u^k = 0. \tag{20}$$

Integrating the square of the polynomial (20)

$$\mathcal{F} = \frac{1}{2}\int_0^u P^2(v)\,\mathrm{d}v = \frac{1}{2}\sum_{k,l=0}^{K} t_{2k+1} t_{2l+1} \frac{(2k+1)!!(2l+1)!!}{k!l!(k+l+1)} u^{k+l+1}, \tag{21}$$

one gets the string free energy—the logarithm of the quasiclassical tau function—due to the formula

$$\mathcal{F} = \frac{1}{2}\sum_{k,l} t_k t_l \, \mathrm{res}_{P_0}(\xi^k \, dH_l), \tag{22}$$

expressing free energy [14] in terms of its second derivatives, and since the coefficient on the rhs of (21) exactly coincides with the second derivative (6),

$$\begin{aligned}
&\mathrm{res}_{\lambda=\infty}(\xi^{2k+1}\, dH_{2l+1}) \\
&\quad = \sum_{n\geqslant 0}\sum_{m=0}^{l} \frac{(2u)^{n+m}}{n!m!} \frac{\Gamma(k+3/2)\Gamma(l+3/2)(2(l-m)+1)}{\Gamma(k+3/2-n)\Gamma(l+3/2-m)} \mathrm{res}_{\lambda=\infty}(d\lambda\, \lambda^{2(k+l-n-m)+1}) \\
&\quad = (2u)^{k+l+1}\Gamma(k+3/2)\Gamma(l+3/2)\frac{2}{\pi}\sum_{m=0}^{l}\frac{(-)^{l-m}}{m!(k+l+1-m)!} \\
&\quad = \frac{(2k+1)!!(2l+1)!!}{k!l!(k+l+1)} u^{k+l+1},
\end{aligned} \tag{23}$$

where the last equality holds, in particular, due to binomial identity $\sum_{m=0}^{l}(-)^m\binom{s}{n} = (-)^l\binom{s-1}{l}$.

3. Examples: particular (p, q) models

3.1. Pure gravity: the explicit partition function

In this case $(p, q) = (2, 3)$, one has only two nontrivial parameters t_1 and t_3, and the partition function can be calculated explicitly. The times (4) are expressed by

$$\begin{aligned}
&t_5 = \tfrac{2}{5}, \qquad t_3 = \tfrac{2}{3}Y_1 - X_0, \qquad t_1 = \tfrac{3}{4}X_0^2 - X_0 Y_1 \\
&t_4 = \tfrac{1}{2}Y_2, \qquad t_2 = Y_0 - Y_2 X_0
\end{aligned} \tag{24}$$

in terms of the coefficients of the polynomials

$$X = \lambda^2 + X_0, \qquad Y = \lambda^3 + Y_2\lambda^2 + Y_1\lambda + Y_0. \tag{25}$$

The odd times t_1, t_3 and t_5 do not depend upon the even coefficients Y_0 and Y_2 of the second polynomial in (25), and in what follows we choose $Y_2 = Y_0 = 0$, ensuring $t_2 = t_4 = 0$. Relations (24) can then be easily solved for the latter coefficients of

$$X = \lambda^2 + X_0, \qquad Y = \lambda^3 + Y_1\lambda, \tag{26}$$

giving rise to

$$X_0 = \tfrac{1}{3}\sqrt{9t_3^2 - 12t_1} - t_3, \qquad Y_1 = \tfrac{1}{2}\sqrt{9t_3^2 - 12t_1}. \tag{27}$$

The second half of residues (4) gives

$$\frac{\partial \mathcal{F}}{\partial t_1} = \frac{1}{8}X_0^3 - \frac{1}{4}Y_1 X_0^2 \qquad \frac{\partial \mathcal{F}}{\partial t_3} = -\frac{1}{8}Y_1 X_0^3 + \frac{3}{64}X_0^4. \tag{28}$$

This results in the following explicit formula for the quasiclassical tau function:

$$\mathcal{F} = \frac{1}{3240}\left(9t_3^2 - 12t_1\right)^{5/2} + \frac{1}{4}t_3^3 t_1 - \frac{1}{4}t_3 t_1^2 - \frac{3}{40}t_3^5. \tag{29}$$

At $t_3 \to \infty$ (expansion at $t_1 \to 0$), formula (29) gives

$$\mathcal{F} \underset{t_3\to\infty}{=} -\frac{t_1^3}{18t_3}\left(1 + O\left(\frac{t_1}{t_3^2}\right)\right), \tag{30}$$

which is the partition function of the Kontsevich model [17, 18] (also identified with the (2, 1)-point or topological gravity). At $t_1 \to \infty$, the tau function (29) scales as $\mathcal{F} \propto t_1^{5/2}$ or as a partition function of the pure two-dimensional gravity; expansion at $t_1 \to \infty$ gives

$$\mathcal{F} = (-3t_1)^{5/2}\left(\frac{4}{405} - \frac{1}{54}\frac{t_3^2}{t_1} + \frac{1}{96}\frac{t_3^4}{t_1^2} + O\left(\frac{t_3^6}{t_1^3}\right)\right) + \cdots \tag{31}$$

modulo analytic terms.

Formula (29) is the only example of *exact* computation. For the rest, one needs to solve perturbatively the nonlinear string equation. It contains a polynomial part, which contributes only to a finite number of correlation functions. Usually such a 'non-universal' part is neglected, when comparing the result of the computation with the worldsheet Liouville theory. It also vanishes at $t_3 = 0$ or at vanishing of the time, corresponding to the so-called boundary operator (see e.g. [21]), the t_{2K-1} variable in the $(2, 2K - 1)$ KdV series, which we shall usually neglect in what follows. However, these terms are essential, when taking the limit (30) to the topological Kontevich model, and it means that they come from the contact terms of topological origin.

3.2. The gravitational Yang–Lee model: (p, q) = (2, 5)

The calculation of times according to (4) gives

$$\begin{aligned} t_1 &= -\tfrac{5}{8}X_0^3 + \tfrac{3}{4}Y_3X_0^2 - Y_1X_0 \qquad & t_3 &= \tfrac{5}{4}X_0^2 - Y_3X_0 + \tfrac{2}{3}Y_1 \\ t_5 &= \tfrac{2}{5}Y_3 - X_0 & t_7 &= \tfrac{2}{7} \end{aligned} \tag{32}$$

for the polynomials

$$X = \lambda^2 + X_0 \qquad Y = \lambda^5 + Y_3\lambda^3 + Y_1\lambda. \tag{33}$$

These equations are easily solved for

$$Y_1 = \tfrac{3}{2}\left(t_3 + \tfrac{5}{4}X_0^2 + \tfrac{5}{2}t_5X_0\right) \qquad Y_3 = \tfrac{5}{2}(X_0 + t_5), \tag{34}$$

ending up with the only nonlinear string equation for X_0:

$$t_1 = -\tfrac{5}{8}X_0^3 - \tfrac{3}{2}t_3X_0. \tag{35}$$

The one-point functions (4) are given by

$$\frac{\partial\mathcal{F}}{\partial t_1} = -\frac{15}{64}X_0^4 - \frac{3}{8}t_3X_0^2, \qquad \frac{\partial\mathcal{F}}{\partial t_3} = -\frac{9}{64}X_0^5 - \frac{3}{16}t_3X_0^3 \tag{36}$$

while the two-point functions are given by

$$\frac{\partial^2\mathcal{F}}{\partial t_1^2} = \frac{X_0}{2}, \qquad \frac{\partial^2\mathcal{F}}{\partial t_1\partial t_3} = \frac{3}{8}X_0^2, \qquad \frac{\partial^2\mathcal{F}}{\partial t_3\partial t_3} = \frac{3}{8}X_0^3. \tag{37}$$

The latter expressions can be obtained by differentiation (36) upon following from (35) explicit formulae for $\frac{\partial X_0}{\partial t_1}$ and $\frac{\partial X_0}{\partial t_3}$, or they follow directly from the Hirota equations (14).

To compare the predictions of the 'integrable' approach for correlators in two-dimensional gravity with the calculations in worldsheet theory, one first needs to make certain correspondences in the space of coupling constants. The simplest one comes from the scaling properties (15) and (16). In the Yang–Lee theory, the role of the cosmological constant is played by the KdV time t_3, and from the scaling properties of the 'fixed area' partition function $F_A(t_1) = A^{-7/2}\mathsf{z}(t_1 A^{3/2})$ (cf [19]) one gets for the Laplace transformed $\mathcal{F}(t_1, t_3) = \int_0^\infty \frac{dA}{A}\, e^{-t_3 A} F_A(t_1)$ or

$$\mathcal{F} = t_3^{7/2}\mathsf{f}\left(\frac{t_1}{t_3^{3/2}}\right) \equiv t_3^{7/2}\mathsf{f}(\mathsf{t})$$
$$\frac{\partial\mathcal{F}}{\partial t_1} = t_3^2\mathsf{f}', \qquad \frac{\partial^2\mathcal{F}}{\partial t_1^2} = t_3^{1/2}\mathsf{f}'', \ldots, \tag{38}$$

and the string equation turns into

$$\mathsf{t} + 5(\mathsf{f}'')^3 + 3\mathsf{f}'' = 0 \tag{39}$$

to be solved for the coefficients $\mathsf{f}_n \equiv \mathsf{f}^{(n)}|_{t=0}$ in the expansion of

$$\mathcal{F} = t_3^{7/2}\mathsf{f}_0 + t_1 t_3^2 \mathsf{f}_1 + \frac{t_1^2 t_3^{1/2}}{2}\mathsf{f}_2 + \frac{t_1^3}{6t_3}\mathsf{f}_3 + \cdots, \tag{40}$$

which gives rise to rational expressions

$$\mathsf{f}_3 = -\frac{1}{3\left(1+5\mathsf{f}_2^2\right)}, \qquad \mathsf{f}_4 = -\frac{10\mathsf{f}_2}{9\left(1+5\mathsf{f}_2^2\right)^3}, \qquad \mathsf{f}_5 = \frac{10\left(1-25\mathsf{f}_2^2\right)}{27\left(1+5\mathsf{f}_2^2\right)^5}$$
$$\mathsf{f}_6 = \frac{1000\mathsf{f}_2\left(1-10\mathsf{f}_2^2\right)}{81\left(1+5\mathsf{f}_2^2\right)^7}, \qquad \mathsf{f}_7 = -\frac{1000\left(1-95\mathsf{f}_2^2+550\mathsf{f}_2^4\right)}{243\left(1+5\mathsf{f}_2^2\right)^9} \tag{41}$$
$$\mathsf{f}_8 = -\frac{70000\mathsf{f}_2\left(2-70\mathsf{f}_2^2+275\mathsf{f}_2^4\right)}{2187\left(1+5\mathsf{f}_2^2\right)^{11}}, \ldots$$

in terms of the two-point function f_2, which itself can be found as a nonvanishing solution to the 'reduced' string equation

$$3\mathsf{f}_2 + 5\mathsf{f}_2^3 = 0. \tag{42}$$

The 'total normalization' f_0 and the 'one-point function' f_1, which does not have a universal sense, since it is coupled to an analytic term in the expansion (40), in principle are determined by the residue formula for $\partial\mathcal{F}/\partial t_3$, or

$$7\mathsf{f} - 3\mathsf{t}\mathsf{f}' + 9(\mathsf{f}'')^5 + 3(\mathsf{f}'')^3 = 0 \tag{43}$$

giving rise to

$$\mathsf{f}_0 = -\tfrac{9}{7}\mathsf{f}_2^5 - \tfrac{3}{7}\mathsf{f}_2^3 = -\tfrac{3}{7}\mathsf{f}_2^3\left(1+3\mathsf{f}_2^2\right) \qquad \mathsf{f}_1 = -\tfrac{9}{4}\mathsf{f}_2^3\mathsf{f}_3 - \tfrac{45}{4}\mathsf{f}_2^4\mathsf{f}_3 = \tfrac{3}{4}\mathsf{f}_2^2. \tag{44}$$

This results in the rational 'invariant ratios', e.g.

$$\frac{\mathsf{f}_4\mathsf{f}_2}{\mathsf{f}_3^2} = -3, \qquad \frac{\mathsf{f}_4\mathsf{f}_3}{\mathsf{f}_2\mathsf{f}_5} = -\frac{1}{8}, \qquad \frac{\mathsf{f}_2\mathsf{f}_4}{\mathsf{f}_0\mathsf{f}_6} = 1, \qquad \frac{\mathsf{f}_4^2}{\mathsf{f}_0\mathsf{f}_8} = -\frac{6}{143},$$
$$\frac{\mathsf{f}_2^2}{\mathsf{f}_0\mathsf{f}_4} = \frac{\mathsf{f}_2\mathsf{f}_6}{\mathsf{f}_4^2} = -35, \tag{45}$$

to be possibly compared with the computations in the worldsheet theory.

3.3. *Mixing in the* $(2, 7)$ *model*

The $(p, q) = (2, 7)$ model is naively not much different from the Yang–Lee case of $(2, 5)$ theory considered in section 3.2. Polynomials (1) are

$$X = \lambda^2 + X_0, \qquad Y = \lambda^7 + \frac{7X_0}{2}\lambda^5 + Y_3\lambda^3 + Y_1\lambda \tag{46}$$

and the calculation of flat times (4) gives

$$t_1 = \tfrac{3}{4}Y_3X_0^2 - \tfrac{105}{64}X_0^4 - Y_1X_0 \qquad t_3 = \tfrac{35}{12}X_0^3 - Y_3X_0 + \tfrac{2}{3}Y_1$$
$$t_5 = -\tfrac{7}{4}X_0^2 + \tfrac{2}{5}Y_3 \qquad t_7 = 0 \qquad t_9 = \tfrac{2}{9}. \tag{47}$$

Again, we see that (47) can be easily solved w.r.t. Y_j, but the only coefficient X_0 now satisfies

$$t_1 = -\tfrac{35}{64}X_0^4 - \tfrac{15}{8}t_5X_0^2 - \tfrac{3}{2}t_3X_0, \tag{48}$$

where we put $t_7 = 0$ for the coefficient at the 'boundary' operator [21].

The one-point functions (4) are given for the (2, 7) model by

$$\begin{aligned}
\frac{\partial\mathcal{F}}{\partial t_1} &= -\frac{7}{32}X_0^5 - \frac{1}{4}Y_1X_0^2 + \frac{1}{8}Y_3X_0^3 = -\frac{7}{32}X_0^5 - \frac{5}{8}X_0^3t_5 - \frac{3}{8}X_0^2t_3\\
\frac{\partial\mathcal{F}}{\partial t_3} &= -\frac{1}{8}Y_1X_0^3 - \frac{35}{512}X_0^6 + \frac{3}{64}Y_3X_0^4 = -\frac{35}{256}X_0^6 - \frac{45}{128}X_0^4t_5 - \frac{3}{16}X_0^3t_3\\
\frac{\partial\mathcal{F}}{\partial t_5} &= -\frac{15}{512}X_0^7 - \frac{5}{64}Y_1X_0^4 + \frac{3}{128}Y_3X_0^5 = -\frac{25}{256}X_0^7 - \frac{15}{64}X_0^5t_5 - \frac{15}{128}X_0^4t_3.
\end{aligned} \tag{49}$$

On the right-hand sides of (49) we already substituted the expressions for Y_j in terms of times (47), and the rest is to solve (48) by expanding in t_3 and t_5 and substitute the result into (49).

The scaling ansatz (16), (38) now reads as

$$\begin{aligned}
&\mathcal{F} = t_5^{9/2}\mathsf{f}\left(\frac{t_1}{t_5^2}, \frac{t_3}{t_5^{3/2}}\right)\\
&\frac{\partial\mathcal{F}}{\partial t_1} = t_5^{5/2}\mathsf{f}^{(1)}, \qquad \frac{\partial^2\mathcal{F}}{\partial t_1^2} = \frac{X_0}{2} = t_5^{1/2}\mathsf{f}^{(11)}, \ldots,
\end{aligned} \tag{50}$$

where we have introduced the shortened notation for the derivatives over the first argument of $\mathsf{f}(\mathsf{t}_1, \mathsf{t}_2)$, and the string equation (48) turns into

$$\mathsf{t}_1 + \tfrac{35}{4}\mathsf{u}^4 + \tfrac{15}{2}\mathsf{u}^2 + 3\mathsf{t}_2\mathsf{u} = 0 \tag{51}$$

for $\mathsf{u} = \mathsf{f}^{(11)}$.

The expansion should be considered in the vicinity of the point $t_1 = \frac{25}{28}t_5^2$, where the one-point function in the first equation of (49) vanishes on the string equation (48) at $t_3 = 0$. It means, in particular, that the function f should be expanded around the non-vanishing background value $\mathsf{t}_1 = \frac{25}{28}$ of its first argument.

3.4. Ising model (p, q) = (3, 4)

The residue formulae for the polynomials

$$X = \lambda^3 + X_1\lambda + X_0, \qquad Y = \lambda^4 + Y_2\lambda^2 + Y_1\lambda + Y_0 \tag{52}$$

give rise to

$$Y_2 = \tfrac{4}{3}X_1 + \tfrac{5}{3}t_5, \qquad Y_0 = \tfrac{2}{9}X_1^2 + \tfrac{10}{9}X_1t_5, \qquad Y_1 = \tfrac{4}{3}X_0 \tag{53}$$

(where the last equation is true upon $t_4 = 0$), while X_0 and X_1 satisfy

$$t_1 = -\tfrac{2}{3}X_0^2 + \tfrac{4}{27}X_1^3 + \tfrac{5}{9}t_5X_1^2, \qquad t_2 = -\tfrac{2}{3}X_0X_1 - \tfrac{5}{3}t_5X_0. \tag{54}$$

Differentiating equations (54), one can find explicitly expressions for the first derivatives

$$\frac{\partial X_1}{\partial t_j} = \frac{Q_1^{(j)}}{R}, \qquad \frac{\partial X_0}{\partial t_j} = \frac{Q_0^{(j)}}{R}, \qquad j = 1, 2, 5, \tag{55}$$

with $R = 4X_1^3 + 12X_0^2 + 20t_5X_1^2 + 25t_5^2X_1$ and

$$\begin{aligned} Q_1^{(1)} &= \tfrac{9}{2}(2X_1 + 5t_5), & Q_0^{(1)} &= -9X_0 \\ Q_1^{(2)} &= -18X_0, & Q_0^{(2)} &= -3X_1(2X_1 + 5t_5) \\ Q_1^{(5)} &= -\tfrac{5}{2}\big(2X_1^3 + 5t_5X_1^2 + 12X_0^2\big), & Q_0^{(5)} &= -5X_1X_0(X_1 + 5t_5). \end{aligned} \tag{56}$$

Solving the second equation of (54) for X_0 and substituting the result into the first one turn it into the Boulatov–Kazakov equation for X_1 [20]:

$$t_1 = -\frac{6t_2^2}{(2X_1 + 5t_5)^2} + \frac{4}{27}X_1^3 + \frac{5}{9}t_5X_1^2 \tag{57}$$

(contains information about all singularities of $\mathcal{F}$ for arbitrary magnetic field t_2 and fermion mass t_5).

It is interesting to compare the Boulatov–Kazakov equation with what gives here formula (19). The branch points are given by $\mathrm{d}X = 0$ for the first polynomial from (52), or $\lambda_\pm = \pm\sqrt{-\frac{X_1}{3}}$, so that the vanishing of the derivative of function

$$S = t_1X(\lambda)_+^{1/3} + t_2X(\lambda)_+^{2/3} + t_5X(\lambda)_+^{5/3} + \tfrac{3}{7}X(\lambda)_+^{7/3} \tag{58}$$

at $\lambda_\pm$ or $S'(\lambda)|_{\lambda_+} = S'(\lambda)|_{\lambda_-}$ gives rise to the last equation of (54), which is to be easily solved for X_0. Substituting the result into $S'(\lambda)|_{\lambda_+} + S'(\lambda)|_{\lambda_-} = 0$ reproduces immediately the string equation (57).

The one-point functions

$$\begin{aligned} \frac{\partial\mathcal{F}}{\partial t_1} &= \frac{1}{27}X_1^4 + \frac{10}{81}t_5X_1^3 - \frac{4}{9}X_1X_0^2 - \frac{5}{9}t_5X_0^2 \\ \frac{\partial\mathcal{F}}{\partial t_2} &= \frac{4}{27}X_1^3X_0 + \frac{10}{27}t_5X_1^2X_0 - \frac{8}{27}X_0^3 \\ \frac{\partial\mathcal{F}}{\partial t_5} &= \frac{40}{243}X_1^3X_0^2 - \frac{10}{2187}X_1^6 + \frac{25}{81}t_5X_1^2X_0^2 - \frac{10}{729}t_5X_1^5 - \frac{5}{27}X_0^4 \end{aligned} \tag{59}$$

give rise to

$$\begin{aligned} &\frac{\partial^2\mathcal{F}}{\partial t_1^2} = \frac{X_1}{3}, \qquad \frac{\partial^2\mathcal{F}}{\partial t_1\partial t_2} = \frac{2X_0}{3}, \\ &\frac{\partial^2\mathcal{F}}{\partial t_1\partial t_5} = \frac{5}{9}X_0^2 - \frac{5}{81}X_1^3, \qquad \frac{\partial^2\mathcal{F}}{\partial t_2\partial t_5} = -\frac{10}{27}X_1^2X_0, \\ &\frac{\partial^2\mathcal{F}}{\partial t_2^2} = -\frac{2}{9}X_1^2, \qquad \frac{\partial^2\mathcal{F}}{\partial t_5^2} = -\frac{50}{81}X_1^2X_0^2 + \frac{5}{243}X_1^5, \ldots. \end{aligned} \tag{60}$$

At $t_2 = 0$ one gets for the one-point functions (59)

$$\begin{aligned} &\left.\frac{\partial\mathcal{F}}{\partial t_1}\right|_{t_2=0} = \frac{1}{27}X_1^4 + \frac{10}{81}t_5X_1^3, \qquad \left.\frac{\partial\mathcal{F}}{\partial t_2}\right|_{t_2=0} = 0 \\ &\left.\frac{\partial\mathcal{F}}{\partial t_5}\right|_{t_2=0} = -\frac{10}{729}X_1^5\left(t_5 + \frac{X_1}{3}\right). \end{aligned} \tag{61}$$

Also note that for $t_2 = 0$ the second equation of (54) has the only reasonable solution $X_0 = 0$, while the first one turns into

$$t_1 = \tfrac{4}{27}X_1^3 + \tfrac{5}{9}t_5X_1^2, \tag{62}$$

which almost coincides with the perturbation of the Yang–Lee $(2, 5)$ model by a quadratic term (cf (35) and note that the potential $\frac{X_1}{3}$ from (62) is an analogue of $\frac{X_0}{2}$ from (35); see (37) and (60). More strictly, the quadratic term can be removed by the redefinition

$$\hat{t}_1 = \tfrac{4}{27}\hat{X}_1^3 - \tfrac{25}{36}t_5^2\hat{X}_1 \qquad \hat{t}_1 = t_1 - \tfrac{125}{216}t_5^3, \qquad \hat{X}_1 = X_1 + \tfrac{5}{4}t_5. \tag{63}$$

This redefinition exactly fits [21] the vanishing of the energy three-point function in the Ising model. Indeed, if one identifies $\hat{t}_1$ with the cosmological constant of the worldsheet theory, the energy three-point functions

$$\frac{\partial^3}{\partial t_5^3}\mathcal{F}\big(\hat{t}_1 + Ct_5^3, t_2 = 0, t_5\big)\bigg|_{t_5=0} = \left(6C\frac{\partial\mathcal{F}}{\partial t_1} + \frac{\partial^3\mathcal{F}}{\partial t_5^3}\right)\bigg|_{t_{2,5}=0} \tag{64}$$

vanish exactly at $C = \frac{125}{216}$. To calculate the rhs of (64) one can use the first equation from (59) and differentiate the last formula from (60) using (55), which is quite easy since $X_0 = 0$ at $t_2 = 0$. An alternative and more fundamental way is to use directly the residue formula (7) for the third derivatives, which gives here

$$\frac{\partial^3\mathcal{F}}{\partial t_5^3} = \mathrm{res}_{\mathrm{d}X=0}\left(\frac{\mathrm{d}H_5^3}{\mathrm{d}X\,\mathrm{d}Y}\right) = \sum_{\lambda=\lambda_\pm}\frac{H_5'(\lambda)^3}{6\lambda Y'(\lambda)} \underset{t_2=t_5=0}{=} -\frac{125}{972}X_1^4. \tag{65}$$

It is interesting to point out that under reparameterization (63) in the space of couplings

$$X_1 = \hat{X}_1 - \tfrac{5}{4}t_5, \qquad t_1 = \hat{t}_1 + \tfrac{125}{216}t_5^3, \tag{66}$$

the reduced string equation (62) acquires the form of (analytically continued) string equation (35) for the Yang–Lee model, with $t_3^{\mathrm{YL}} \sim t_5^2$ of the Yang–Lee model being substituted by the *square* of $t_5 = t_5^{\mathrm{Ising}}$ of the (reduced) Ising model. However, one should use the scaling ansatz (15) rather than (16), as has been used for the $(2, 5)$ theory in (38) for the function $F(\hat{t}_1, t_5) = \mathcal{F}\big(\hat{t}_1 + \frac{125}{216}t_5^3, t_5\big)\big|_{t_2=0}$. Since

$$\frac{\partial^2 F}{\partial\hat{t}_1^2} = \hat{X}_1\big(\hat{t}_1, t_5^2\big) - \frac{5}{4}t_5 \tag{67}$$

as follows from the string equation (63) for $\hat{X}_1$ and the couplings are dimensional, the gravitational Ising free energy $F = \hat{F}\big(\hat{t}_1, t_5^2\big) - \frac{5}{8}t_5\hat{t}_1^2$ is an even function of t_5, apart from an analytic cubic term, and its expansion gives all the $\langle\epsilon^{2n}\rangle$ correlators of the gravitationally dressed $(3, 4)$ Ising model. We shall comment more about the relation of these two models in the following section.

4. Ising versus Yang–Lee

Both gravitational Ising model and $(2, 5)$ Yang–Lee minimal theory arise as two different critical points in a system of Ising spins on a random lattice. Moreover, since both theories have $p + q = 7$, they have identical scaling in the first KP variable (15), which originally has caused a confusion, when distinguishing these two minimal string theories. In particular, this originates from the fact that the string equations of these two models can be obtained from each other by simple reparameterization in the space of couplings, as we have already noted in the previous section.

However, the physical sense of parameters, arising in these two equations, is totally different. One can say that the same KP time variable has different 'quantum numbers', when one takes a solution, corresponding to a different critical point. For example, the role of a cosmological constant, coupled to a unity operator on the worldsheet, is played by t_1 in the $(3, 4)$ Ising theory, but by t_3 in the $(2, 5)$ Yang–Lee theory. Below, we shall try to present more details about this relation and describe it as much as possible from the point of view of the (dispersionless) KP theory.

4.1. Kostov equation

The Kostov equation is a name given by Alesha Zamolodchikov to a 'phenomenological' transcendental equation, satisfied by the second derivative of free energy over the cosmological constant $u \sim \frac{\partial^2 \mathcal{F}}{\partial x^2}$ of the form

$$u^{\nu} + t u^{\nu-1} = x, \tag{68}$$

where $\nu = \nu(p, q) = \frac{p}{q-p}$. For the cases of interest, one gets integer $\nu(2, 3) = 2$ for pure gravity and $\nu(3, 4) = 3$ for Ising (both are unitary with $q = p + 1$), but $\nu(2, 5) = \frac{2}{3}$.

Hence, for the Ising model the Kostov equation reads

$$u_{\mathrm{I}}^3 + t_{\mathrm{I}} u_{\mathrm{I}}^2 = x_{\mathrm{I}} \tag{69}$$

and coincides (after renormalization $x_{\mathrm{I}} \sim t_1, t_{\mathrm{I}} \sim t_5$ and $u_{\mathrm{I}} \sim X_1 = 3\frac{\partial^2 \mathcal{F}}{\partial t_1^2}$; below in this section, we shall use different normalizations from the conventional ones of KP theory, to get rid of ugly numerical constants) with the Boulatov–Kazakov equation (62) when $t_2 = 0$, i.e. for a vanishing magnetic field.

For the Yang–Lee model equation (68) $u_{\mathrm{YL}}^{2/3} + t_{\mathrm{YL}} u_{\mathrm{YL}}^{-1/3} = x_{\mathrm{YL}}$ after the substitution $u_{\mathrm{YL}} = v_{\mathrm{YL}}^3$ turns into

$$v_{\mathrm{YL}}^3 - x_{\mathrm{YL}} v_{\mathrm{YL}} = -t_{\mathrm{YL}} \tag{70}$$

which coincides (again, up to a similar renormalization of couplings) with the Yang–Lee string equation (35) upon $t_3 \sim x_{\mathrm{YL}}, t_1 \sim t_{\mathrm{YL}}$ and $X_0 = 2\frac{\partial^2 \mathcal{F}}{\partial t_1^2} \sim v_{\mathrm{YL}}$.

Comparing (70) with (69) one finds that, as we have already done in the previous section, one may indeed identify u_{I} with v_{YL} after appropriate shifts of the variables (66) and point out the change of the quantum numbers: $t_1 \sim x_{\mathrm{I}} \sim t_{\mathrm{YL}}$ and $t_3 \sim t_{\mathrm{I}}^2 \sim x_{\mathrm{YL}}$.

The relation $u_{\mathrm{YL}} \sim v_{\mathrm{YL}}^3$ is quite clear from the point of view of equations (14). It is just a particular Hirota equation for the dispersionless KP hierarchy, expressing

$$u_{\mathrm{YL}} \sim \frac{\partial^2 \mathcal{F}}{\partial t_3^2} = 3\left(\frac{\partial^2 \mathcal{F}}{\partial t_1^2}\right)^3 \sim v_{\mathrm{YL}}^3 \tag{71}$$

as the function satisfying equation (68), and being here a double derivative of free energy w.r.t the *third* time of the hierarchy, in terms of the canonical KP potential, it being always a double derivative w.r.t. the first time.

From the point of view of the KP theory it is also rather clear why equation (68) is applicable only for $p < q < 2p$, in particular only for $K = 2, 3$ with $\nu(2, 2K - 1) = \frac{2}{2K-3}$. When transforming it to a conventional KdV string equation (20), as was done in (70) for the Yang–Lee model, one finds that the variable t should be generally identified with the t_{7-2K}th time of the KP hierarchy, which does not have a clear sense at $K > 3$.

4.2. Zamolodchikov curve for Ising

If all parameters of the gravitational Ising model are 'alive', the best way is to study, following [19], the fixed area partition function

$$Z_a = \int \frac{\mathrm{d}x}{2\pi \mathrm{i}a} u\, \mathrm{e}^{xa} = -\int \frac{\mathrm{d}u}{2\pi \mathrm{i}a^2}\, \mathrm{e}^{xa}$$
$$x = u^3 + \frac{3}{2}Tu^2 + \frac{H^2}{(u+T)^2}, \tag{72}$$

where we again use the rescaled variables $x = x_{\mathrm{I}} \sim t_1, H \sim t_2, T \sim t_5$ and the rescaled Boulatov–Kazakov equation (57) for $u \sim \frac{\partial^2 \mathcal{F}}{\partial x^2}$. The saddle point equation $\frac{\mathrm{d}x}{\mathrm{d}u} = 0$ for the integral in (72) is given by

$$u(u+T)^4 = \tfrac{2}{3}H^2 \equiv \xi^2 T^5. \tag{73}$$

In rescaled variables $u \sim TU$, the saddle point equation (73) presents a Riemann surface

$$U(U+1)^4 = \xi^2, \tag{74}$$

which is a double-cover of the U_c-plane and a five-sheet cover of the 'magnetic' plane ξ, and the function x on the curve (74) contains a description of all singularities in the gravitational Ising model [23].

In particular, the Yang–Lee singularity arises at a critical value of the magnetic field $H = H_c$, where two values $\xi_c = \sqrt{\frac{2}{3}}\frac{H_c}{T^{5/2}} = \pm \mathrm{i}\frac{16}{5^{5/2}}$ correspond to two points on the curve (74)

$$U_c = -\frac{1}{5}, \qquad \xi_c^2 = U_c(U_c+1)^4 = -\frac{4^4}{5^5}, \tag{75}$$

where $\frac{\mathrm{d}\xi}{\mathrm{d}U}\big|_{U=U_c} = 0$. At the Yang–Lee point, one also obviously has $\frac{\mathrm{d}^2 x}{\mathrm{d}U^2}\big|_{U=U_c} = 0$, and

$$x = -\frac{7}{50}T^3 + \frac{5}{2}(U-U_c)^3 + \cdots = x_c + \frac{625}{128}T^3\left(\frac{\xi^2 - \xi_c^2}{2}\right)^{3/2} + \cdots \tag{76}$$

so that $X \sim \frac{x - x_c}{T^3} \sim \mu^{3/2}$ scales as the right fractional power of the cosmological constant $x_{\mathrm{YL}} = \mu \sim \xi - \xi_c$ in the Yang–Lee model, corresponding to the well-known scaling $t_1 \sim t_3^{3/2}$ of KP times at the critical point with $p + q = 7$. For the expansion of the Boulatov–Kazakov equation, one can now write

$$\frac{X}{\epsilon^3} \sim \mu V + V^3 + O(\epsilon), \tag{77}$$

where $H - H_c \sim \epsilon^2 \mu, U + \frac{1}{5} \sim \epsilon V$, i.e. the Zamolodchikov curve in the vicinity of the Yang–Lee singularity is described, up to renormalization of parameters, by the string equation (35).

5. Discussion

We have tried to demonstrate in this paper that all spherical correlation functions in the quantum Liouville gravity are contained and can be easily extracted from the 'science of polynomials'—dispersionless KP hierarchy. A simple collection of residue formulae allows us to extract the invariant ratios, to be further compared with the correlation functions in worldsheet theory, which can now also be computed—though in a much more cumbersome way—mostly due to the results of Alesha Zamolodchikov in the Liouville theory.

Such an application of classical integrable science to the problems of two-dimensional quantum physics is already a step towards dynamical physics from topological strings, where similar science has been already used with visible success; see e.g. [17, 18, 25–28].

The most nontrivial point in the application of this 'integrable science' is its interpretation in terms of the worldsheet theory. The first point concerns resonances [21, 22], which allow for nonlinear relations between the couplings in KP and worldsheet theories when the fractions of the KPZ scaling dimensions of couplings [2] are an integer. We have observed this phenomenon in the cases of $(2, 7)$ and $(3, 4)$ minimal theories, where this phenomenon is quite easy to take into account by just playing with the residue formulae for one-, two- and three-point functions. Naively, these resonance reparameterizations look like particular W-flows in the space of couplings, but this question deserves further investigation.

Another nontrivial point is related to the fact that KP times may have a different physical sense, or different quantum numbers, when we expand a KP solution around different backgrounds, corresponding to particular minimal theories. We have noted this, say, when comparing the formulations for the gravitational Ising and Yang–Lee models. This is the simplest observation for a very important generic fact that physical observables may change their quantum numbers, when effective field theory is moved in the moduli space; for example, in four-dimensional supersymmetric gauge theories, electrically charged objects may capture magnetic charges and vice versa. Two-dimensional quantum gravity is therefore a good laboratory for studying such effects.

Finally, let us say a few words on how the picture of dispersionless KP for the minimal string theory could be deformed towards quasiclassical hierarchies of a generic nonsingular type. An invariant way to look at the basic polynomials $X = \lambda^p + \cdots$ and $Y = \lambda^q + \cdots$ (1) is to say that they satisfy an algebraic equation

$$Y^p - X^q - \sum f_{ij} X^i Y^j = 0 \tag{78}$$

with some particular coefficients $\{f_{ij}\}$. Generally, for arbitrary coefficients this is a smooth curve of genus

$$g = \frac{(p-1)(q-1)}{2}, \tag{79}$$

which is a resolution or desingularization of the situation, when X and Y can be parameterized as a polynomial of a uniformizing global variable λ. This number coincides with the number of primaries in the corresponding minimal conformal (p, q) theory. Such curves can be obtained, say, by reduction of the curve of the two-matrix model [24]. An interesting example is the hyperelliptic curve of the $(2, 7)$ model $Y^2 = X^7 + \sum_{j=0}^{5} f_j X^j$, satisfied by (46) with the coefficients $f_j = f_j(X_0, Y_1, Y_3)$, $j = 0, \ldots, 5$. At vanishing times (except for $t_9 = 2/9$, see (47)), it shrinks to a cusp $Y^2 = X^7$ or $Y^2 = X^3\big(X^2 + \frac{5t_5}{2}\big)^2$ for nonvanishing cosmological t_5, which is however 'resolved' by passing to worldsheet times $t_1 \to t_1 + \frac{25}{28}t_5^2$ as $Y^2 = (X-u)\big(X^3 + \frac{u}{2}X^2 - \frac{u^2}{2}X - \frac{u^3}{8}\big)^2$ with $u^2 = -\frac{20t_5}{7}$. This form directly generalizes the curve of the Yang–Lee model $Y^2 = (X-u)\big(X^2 - \frac{u}{2}X + \frac{u^2}{4}\big)^2$ for (33) with $u^2 = -\frac{12t_3}{5}$ being proportional to nonvanishing cosmological time.

For such curves, the residue formulae we have discussed above should be extended by period integrals $\oint Y \, \mathrm{d}X$ along all nontrivial cycles on the curve (78). The sense of such period integrals is analogous to the Seiberg–Witten periods or the filling fractions in the matrix models. As usual in quasiclassical hierarchies, the appearance of corresponding period variables reflects an increasing number of unfrozen coefficients in equation (78) or new deformations of the background of the minimal string theories. The study of such deformations is again a long-standing, but still an open, problem.

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